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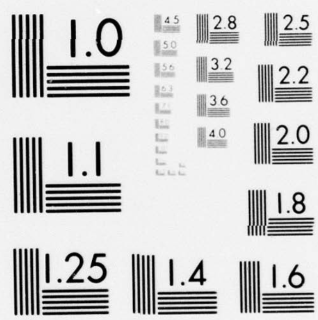

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The question of the continuous dependence of solutions of the hydrodynamic free boundary problem on the data (stability) is investigated. The analysis is for solutions of a generalized hydrodynamic algorithm presented elsewhere. There is seen to be a close connection between the existence of suitable stability results and the principle of energy conservation. Flows are defined to be turbulent or non-turbulent according to the status of their energy conservation. A stochastic framework for analyzing problems of stability is proposed, and a model problem of the stochastic evolution of solutions of an ordinary differential equation is presented.

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STABILITY, ENERGY CONSERVATION, AND TURBULENCE FOR WATER WAVES

by

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1. Introduction

In our last lecture (Ref. 7) we proposed a description of incompressible flow as the evolution of hyperbolic conservation laws subject to a constraint. Representing the time-unfolding of the flow as the action of a nonlinear semigroup $S^*(t)$ on the initial data, we gave a specific characterization of the flow by offering an approximate expression for $S^*(t/n)$ -- we called it $\bar{S}(t/n)$ -- and requiring that $S^*(t)$ be the limit as $n \to \infty$ of $(\bar{S}(t/n))^n$ in some appropriate function space.

We hasten to point out that we have not proven convergence of $(\bar{S}(t/n))^n$ as $n \to \infty$. A central purpose of this paper will be to begin this task and see, in a qualitative manner, what sorts of phenomena we may expect to arise and what special difficulties we should anticipate. For ease of reference, we present here the governing equations, first for classical flows, and then for the generalized flows considered in the last lecture.

The classical initial value problem for the flow, under gravity, of an inviscid incompressible fluid with a free surface but no rigid boundaries, except for a rigid boundary at $z \to -\infty$, requires that one find $u(x,t)$, $P(x,t)$, $\mathcal{A}(t)$ satisfying

$$u_t + u \cdot \nabla u = -\frac{1}{\rho_0} \nabla P - g\vec{k} \quad , \ x \in \mathcal{A}(t),\ 0 < t < T, \qquad (1.1a)$$

$$\nabla \cdot u = 0 \quad , \quad x \in \mathcal{A}(t),\ 0 < t < T, \qquad (1.1b)$$

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subject to initial conditions

$$u(x,0) = u_0, \tag{1.2a}$$

$$\mathcal{A}(0) = \mathcal{A}_0 , \tag{1.2b}$$

and boundary conditions

$$P \to -\rho_0 gz \text{ as } z \to -\infty, \tag{1.3a}$$

$$P = 0 \text{ on } \partial\mathcal{A}(t),$$

$$V \cdot n = u \cdot n \text{ on } \partial\mathcal{A}(t), \tag{1.3c}$$

where $V \cdot n$ is the outward normal velocity of $\mathcal{A}(t)$ (Ref. 4).

In the generalized hydrodynamics (Ref. 7) we have replaced (1.1) - (1.3) by the system of hyperbolic conservation laws

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.4a}$$

$$(\rho u)_t + \nabla \cdot (\rho uu) = -\rho g\vec{k} , \tag{1.4b}$$

subject to the constraint

$$\rho \le \rho_0 , \tag{1.4c}$$

the initial conditions

$$u(x,0) = u_0 , \tag{1.5a}$$

$$\rho(x,0) = \rho_0(x) , \tag{1.5b}$$

and the boundary conditions

$$u \cdot \vec{k} \to 0 \text{ as } z \to -\infty \tag{1.6}$$

The algorithm for calculating $\bar{S}(\tau)$ proceeds in two steps. In the first step one "solves" (1.4a) and (1.4b) for a time τ . The solution of these conservation laws is related to the motion in a gravitational field of a system of particles which collide inelastically (Ref. 7). In the second part (1.4c) is satisfied by solving a one-phase Stefan problem. In trying to prove convergence of $(\bar{S}(t/n))^n$ to $S^*(t)$, we hope to show, in some sense, that

$$\bar{S}(\tau) - S^*(\tau) = o(\tau). \tag{1.7}$$

In that case, evolution of a system under the discrete "semi-group" $(\bar{S}(\tau))^i$ may be thought of as evolution under the limiting semi-group $(S^*(\tau))^i$, with the addition of a "perturbation". Accordingly it is of interest, in

approaching the question of convergence, to inquire about the stability of the basic flow S*(t) to a perturbation. Throughout this paper, by "stability" we mean stability in the sense of Hadamard, that is, continuous dependence of the solution, in an appropriate function space, on the parameters of the problem.

We will begin by presenting stability results already obtained for the flow as classically formulated in (1.1) - (1.3) (Ref. 6). Subsequent analysis will be motivated in part by a need to carry over the classical stability results to the generalized flows satisfying (1.4) - (1.6). The classical Helmholtz instability will assume particular importance for the generalized flows. In the process of transcribing our stability results from a form appropriate to (1.1) - (1.3) to a form more in harmony with the algorithm offered for the approximate solution of (1.4) - (1.6), we will observe a relationship that exists between stability results for flows and energy conservation for the same flows. Exploiting this relationship and using some ab initio ideas of ours relating instability to turbulence, we will next define the turbulence or absence thereof of a flow in terms of its energy-preserving properties. Then our focus will shift from hydrodynamics, as expressed in (1.4) - (1.6), to the hyperbolic conservation laws (1.4a,b), since our analysis to that point will have indicated that the origins of Helmholtz instability, energy dissipation, and turbulence are to be found in the solution of these conservation laws. On determining that the initial value problem for the conservation laws fails the Hadamard test for well-posedness in the function spaces appropriate to the stability results, we will suggest a weaker form of Hadamard's criterion. This will cause us to enlarge the realm of our solutions and perturbations to the stochastic domain. Function spaces which seem appropriate for treatment of the conservation laws will be described. A final section will describe qualitatively some of the results we expect, and the stochastic solution of an initial value problem for an ordinary differential equation which is formulated deterministically will illustrate our point.

Perhaps the most emotionally charged of the words we have used is "turbulence". We shall devote the rest of this section to an explanation of the reasons for our particular approach to the subject, and the respects in which it differs from or resembles other approaches. Our approach has been to regard the equations (1.4) (or just (1.4a,b)) fundamentally in a deterministic sense. This puts us at variance at the outset with the approach which is statistical throughout. We acknowledge that experimentally turbulent phenomena exhibit statistical behavior, but an approach which is completely statistical seems to be beset with problems, such as the closure problem, which do not arise in the deterministic picture. The statistical picture seems very useful for interpreting what one often measures in an experimental situation, and for providing a general set of

guidelines or framework for correlating experimental data, but we have not been convinced of its usefulness for providing an understanding of turbulence. Moreover, our goal is not to develop a theory of turbulence as such, but to provide a consistent description of the solutions of the generalized hydrodynamic equations (1.4) - (1.6). Our motivation is our belief that the theory cannot be completed without consideration of those flows which we shall define in the sequel to be "turbulent". To us a more attractive course than the introduction of statistics at the very beginning is the collection of as much information as we can obtain about solutions of the basic flow equations, and then perhaps the incorporation of our results into an ensemble picture. Even in the statistical theories, one usually assumes the validity of the underlying Navier-Stokes or Euler equations. The fact that ultimately we seem to be forced to regard (1.4) or (1.4a,b) as determining a Markov process rather than deterministic semi-group may be of some comfort to workers in the statistical theories and may offer some promise of experimental verification of our theory in the future, but it certainly does not vitiate our approach.

A problem common to some theories of turbulence seems to be their dependence on various scales of length, time, etc. Thus, the same phenomenon may be viewed as "laminar" or "turbulent" according to the perspective of the viewer. There is nothing wrong in practice with regarding some quantities as "large" and others as "small", with keeping some terms in equations and dropping others in order to achieve approximate solutions. But we should recognize these characterizations of phenomena for what they are: statements of an asymptotic nature about the phenomena studied. It seems to us that the word "turbulence", if it has any meaning at all, is perhaps a description of an asymptotic state of events, and that any concise and mathematically complete description of turbulence should describe this asymptotic state, without reference to scales. In reality, this asymptotic state may only be an abstraction and may never be achieved and yet, like many asymptotic notions, it may be worthy of study as a distillation of the essence of the phenomena about which understanding is sought.

The hydrodynamic stability approach to turbulence seems to conform to this asymptotic characterization. One looks for regions of the parameter space where perturbations grow in time. On the assumption that the lapsed time is infinitely large, one associates such regions of growth with the appearance of large-scale deformations in the flow due to perturbations, no matter how small, and with the onset of turbulence. Closely connected is the study of time-independent (late-time) behavior of a system as parameters of the system, such as Reynolds' number, vary. In this case, at certain critical values of Reynolds' number, the late-time solutions may split apart, or bifurcate. In both these views, the asymptotic parameter is the time.

In our approach, the asymptotic parameter is Reynolds' number Re, which becomes infinite as kinematic visosity $\nu \to 0$. Whereas in the hydrodynamic stability approach one looks for finite temporal growth rates, which amplify perturbations over an infinitely long time, in our theory we look for Hadamard instabilities, which magnify perturbations infinitely in a finite time. Thus, our approach is essentially a time-dependent one, and the turbulence or non-turbulence of the flow may change as time progresses. Real flows have viscosity of course and, although it hasn't been proven, it is reasonable to believe that such flows will have global solutions in time with a certain degree of regularity, for any $\nu > 0$. Nevertheless, in practice the length scale on which regularity exists may be extremely small (and here is where we make the transition to asymptotics), and one may want to refer to the appropriate inviscid equations to describe the flow in question. Thus, in our approach, one would proceed from $Re = \infty$ down, in contrast to the hydrodynamic stability approach, where one often proceeds from $Re = 0$ up. For Re very large, our approach seems more desirable to us. Nevertheless, the different apporaches may not be in fundamental conflict, but may instead be complementary.

In closing this section, we note that there is still much more that we do not understand about this subject than that we do. Nevertheless, it seems reasonably certain to us that a number of our ideas will remain unaltered by further research. One of these is the crucial connection between energy conservation and turbulence. And enough parts of the puzzle have fallen together that we are confident others will fall in place. Turbulence, as we perceive it, involves some fairly deep properties of the Hamiltonian mechanics of systems with infinitely many particles.

2. Stability of Flows

We begin with a stability result found previously (Ref. 6) for solutions to (1.1) - (1.3). This is obtained by comparing the solution $(u, P, \mathcal{A})$ to (1.1) - (1.3) with the solution $(\tilde{u}, \tilde{P}, \tilde{\mathcal{A}})$ to the same equations, except that u_0 and $\mathcal{A}_0$ in (1.2) are replaced by $\tilde{u}_0$ and $\tilde{\mathcal{A}}_0$, respectively. Let us write $\tilde{u} = u + u'$ and define ζ' as the distance one travels from $\partial\mathcal{A}$ outward along a normal to $\partial\mathcal{A}$ to get to $\partial\tilde{\mathcal{A}}$. The stability relation is

$$\frac{d}{dt}\left| \int_{\mathcal{A}(t)} \rho_0 \tfrac{1}{2}(u')^2\, dx + \int_{\partial\mathcal{A}(t)} \tfrac{1}{2}(\zeta')^2 \left(-\frac{\partial P}{\partial n}\right) dS \right| \tag{2.1}$$

$$= - \int_{\mathcal{A}} \rho_0\, u' \cdot (u' \cdot \nabla)\, u dx + \int_{\partial\mathcal{A}} \tfrac{1}{2}(\zeta')^2 \left| \frac{d}{dt} \left(-\frac{\partial P}{\partial n}\right) + n \cdot (n \cdot \nabla) u \left(-\frac{\partial P}{\partial n}\right) \right| dS$$

Here n is a unit vector in the direction of the outward normal to $\partial\mathcal{A}$ and

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla \tag{2.2}$$

This is a stability relation because $P \geq 0$ in the water -- one would get cavitation before achieving $P < 0$ inside. With (1.3b), we get $-\frac{\partial P}{\partial n} \geq 0$ on $\partial\mathcal{R}(t)$. Thus the case of Taylor instability, where $-\frac{\partial P}{\partial n} < 0$ on $\partial\mathcal{R}(t)$, does not arise in problems of interest to us here. See Reference 6 for a more complete discussion. ((2.1) is actually obtained from the linearized equations for the perturbations u' and ζ', and hence is only valid to the lowest order in the perturbed quantities. A stability relation which is valid for general perturbations u' and ζ' can be obtained with a little more effort. The more rigorous result looks very much like (2.1), and the essence of the stability theory lies in the result obtained for the linearized perturbation equations (Ref. 6).)

The equations (1.4) - (1.6) for a generalized flow make no reference to the boundary $\partial\mathcal{R}$. Accordingly, we expect the explicit boundary terms to disappear from any sufficiently general stability relation for non-classical flows. Therefore, let us consider for a moment the flow of a barotropic fluid, for which no sharp density interface appears. The governing equations for the flow (u, P, ρ) in a region $\mathcal{R}(t)$ moving with the fluid are

$$\rho_t + \nabla\cdot(\rho u) = 0 \tag{2.3a}$$

$$(\rho u)_t + \nabla\cdot(\rho uu) = -\rho g\vec{k} - \nabla P , \tag{2.3b}$$

$$P = P(\rho), \tag{2.3c}$$

and initial and boundary conditions are the same as (1.5) and (1.6). Comparing (2.3) with the equations for a perturbed flow $(\tilde{u}, \tilde{P}, \tilde{\rho})$ and letting

$$\tilde{u} = u + u', \tag{2.4a}$$

$$\tilde{P} = P + P', \tag{2.4b}$$

$$\tilde{\rho} = \rho + \rho', \tag{2.4c}$$

we find, to first order in the perturbations,

$$\rho' t + \nabla\cdot(\rho' u + \rho u') = 0 , \tag{2.5a}$$

$$u'_t + u \cdot \nabla u' + u' \cdot \nabla u = -\frac{\nabla P'}{\rho} + \frac{\rho' \nabla P}{\rho^2} , \tag{2.5b}$$

$$P' = \frac{dP}{d\rho}\rho' , \tag{2.5c}$$

and

$$u'(x,0) = u'_0, \tag{2.6a}$$

$$\rho'(x,0) = \rho'_0, \tag{2.6b}$$

$$u' \cdot\vec{k} \to 0 \text{ as } z \to -\infty. \tag{2.6c}$$

Multiplying (2.5b) by $\rho u'$, multiplying (2.5a) by $\frac{\rho'}{\rho}\frac{dP}{d\rho}$, and adding, we get

$$\frac{d}{dt}\int_{\mathcal{R}(t)} [\tfrac{1}{2}\rho(u')^2 + \frac{1}{2\rho}P'\rho']dx$$

$$=\int_{\mathcal{R}}\left[\rho u'\cdot(u'\cdot\nabla)u + \frac{(\rho')^2}{2\rho}\ \frac{d}{dt}\left(\frac{dP}{d\rho}\right)\right] dx - \int_{\partial\mathcal{R}} P'u'\cdot n dS\ . \tag{2.7}$$

Since $P' = \frac{dP}{d\rho}\rho'$ and $\frac{dP}{d\rho} \geq 0$ for real fluids, (2.7) is a stability relation.

If we have

$$P(\rho) = \begin{cases} 0 & \rho \leq \rho_o \\ \dfrac{\rho-\rho_o}{\varepsilon} & \rho \geq \rho_o\ , \end{cases} \tag{2.8}$$

we get the equation of state for an incompressible fluid in the limit as $\varepsilon \to 0$. In this limit the perturbation quantities on the left-hand side of (2.7) become

$$\rho' \to \rho_o\ \delta_{\partial\mathcal{R}_f}\ \zeta' \tag{2.9a}$$

and

$$P' \to -\ \frac{\partial P}{\partial n}\ \zeta' \tag{2.9b}$$

where $\partial\mathcal{R}_f$ is the set of points where ρ is discontinuous and $\delta_{\partial\mathcal{R}_f}$ is the Dirac measure concentrated on $\partial\mathcal{R}_f$. With the limits (2.9), equation (2.7) reverts to the stability equation (2.1) for water waves. In this sense we may regard the water wave problem as a singular transonic flow problem, in which the water surface corresponds to the sonic line, and the Mach number is infinite on one side of the sonic line and zero on the other.

The terms on the left of (2.7) comprise a positive definite functional of the perturbation, and we may think of them as representing the "energy" of the perturbation. The analogy with an energy is so suggestive that we explore the matter further.

The statement of energy conservation for the classical flow satisfying (2.3) is

$$\varepsilon_t + \nabla\cdot(u\varepsilon) = -\nabla\cdot(uP), \tag{2.10a}$$

where

$$\varepsilon(\rho,u) = \tfrac{1}{2}\rho u^2 + \rho g z + \rho\int\frac{dP}{\rho} - P. \tag{2.10b}$$

This is easily obtained from (2.3). The integrated form of (2.10) is

$$\frac{dE}{dt} = -\int_{\partial\mathcal{R}} Pu\cdot n dS\ , \tag{2.11a}$$

$$E = \int \varepsilon dx \tag{2.11b}$$

Now, suppose at a given time t_o we have a basic flow (u,P,ρ) and a perturbed flow

$$(u^1,\ P^{(1)},\ \rho^{(1)} = (u + u',\ P^{(1)},\ \rho + \rho'). \tag{2.12a}$$

The flow $(u^{(1)}, P^{(1)}, \rho^{(1)})$ satisfies equations like (2.3), and (2.12a) may then be used to define u' and ρ' for $t > t_o$. Consider also the perturbed flows whose fields are given, at the specified time t_0, by

$$(u^{(2)},\ P^{(2)},\ \rho^{(2)}) = (u - u',\ P^{(1)},\ \rho + \rho'), \tag{2.12b}$$

$$(u^{(3)},\ P^{(3)},\ \rho^{(3)}) = (u + u',\ P^{(3)},\ \rho - \rho'), \tag{2.12c}$$

and

$$(u^{(4)},\ P^{(4)},\ \rho^{(4)}) = (u - u',\ P^{(3)},\ \rho - \rho'), \tag{2.12d}$$

Each one of the perturbed flows satisfies an energy balance equation like (2.10):

$$(\varepsilon(\rho^{(i)},\ u^{(i)}))_t + \nabla \cdot (u^{(i)}\varepsilon(\rho^{(i)},\ u^{(i)})) = -\nabla \cdot (u^{(i)}P^{(i)}),\ 1 \le i \le 4. \tag{2.13}$$

In addition, it satisifes evolution equations like (2.3). We may expand the terms in (2.13) out to second order in the perturbations ρ' and u', taking care to express $\rho_t^{(i)}$ and $u_t^{(i)}$ for i = 2,3,4 in terms of ρ'_t, u'_t, and second order expressions in ρ' and u' at $t = t_0$. If we now add up the energy balance equations for the four perturbed flows, subtract off four times the energy balance equation for the basic flow, and integrate over space, we will get the stability relation (2.7) at t_0. (This procedure may be followed more generally to go from an energy conservation equation like (2.10a) to a stability relation like (2.7). For the result to be a stability relation, all we require from ε(ρ,u) in (2.10b) is that

$$\frac{\partial^2 \varepsilon}{\partial \rho^2} \ge 0,\ \frac{\partial^2 \varepsilon}{\partial u_i \partial u_j} \text{ positive definite.} \tag{2.14)}$$

For sufficiently smooth basic flows, (2.1) and (2.7) enable one to bound the positive definite quadratic functionals on the left-hand sides of the respective equations by their initial values multiplied by an exponential function of the time. For example, suppose that the velocity is differentiable, $|\nabla u| \le K$ for a constant K. Then the term $\int \rho u' \cdot (u' \cdot \nabla) u\, dx$ which appears on the right-hand side of both (2.1) and (2.7) is bounded by $K \int \rho (u')^2 dx$. This term arises independently of the equation of state and is closely connected with the stability of the solution of the hyperbolic conservation laws (1.4a,b). The other terms on the right-hand sides of (2.1) and (2.7) are associated with the respective equations of state for the flows. When $\frac{d^2P}{d\rho^2}$ is bounded and $|\nabla u| \le K$, the term

$$\int \frac{(\rho')^2}{2\rho} \frac{d}{dt}\left(\frac{dP}{d\rho}\right) dx$$

in (2.7) may be bounded as follows:

$$\int \frac{(\rho')^2}{2\rho} \frac{d}{dt}\left(\frac{dP}{d\rho}\right) dx = \int -\frac{(\rho')^2}{2} \frac{d^2P}{d\rho^2} \nabla \cdot u\, dx \le \frac{K}{2} \int (\rho')^2 \left|\frac{d^2P}{d\rho^2}\right| dx \qquad (2.15)$$

Thus, for $\frac{d^2P}{d\rho^2}$ bounded, the only source of instability is the potential lack of Lipschitz continuity in the velocity field. In the incompressible case, in the absence of a free boundary, the only term on the right-hand side of (2.1) is that associated with the conservation laws, and thus the equation of state is not in itself a direct source of instability for the liquid. Likewise, our anticipation is that the equation of state (1.4c) will not be a source of instability in the interior of the liquid region for general incompressible flows.

As we have pointed out, we are interested in non-classical flows which satisfy equations (1.4) - (1.6). For generalized flows, certainly for those with a free boundary, we do not expect the velocity field to be uniformly Lipschitz continuous (Ref. 7). Since Lipschitz continuity was so important in proving stability for classical flows, it seems quite possible that solutions of the hyperbolic conservation laws will not be stable in the general case. Accordingly, if we are to correlate a lack of stability with turbulence, it appears that, in our split-step scheme for solving (1.4) - (1.6), the onset of turbulence will at least manifest itself in the solution of the conservation laws, and that we are justified in focusing our attention on this part of the algorithm, instead of the one-phase Stefan problem, in order to obtain some initial insight into the nature of turbulence, as well as some insight into the sense in which the initial value problem is well-posed.

3. Relation of Energy Conservation to Turbulence

Instability, which we associate with turbulence, has been seen to have a connection with the principle of energy conservation for a flow. In addition, it appears to be related to a loss of Lipschitz continuity for the velocity field. We have dwelt in the past on the fact that our procedure for solving (1.4) - (1.6) allows the dissipation of energy through the inelastic collision of fluid elements. Such inelastic collision can only occur when the velocity field is not Lipschitz continuous.

It has, of course, been apparent all along that energy-dissipating solutions are not classical solutions, since energy conservation is an integral part of the classical inviscid theory. In addition, energy-dissipating solutions are not reversible. We noted in the last lecture (Ref. 7) that the velocity is generally discontinuous when a wave spills over and impacts the remainder of the surface. In such a case we expect energy dissipation, in accordance with the discussion of the example described by Figure 1 of the last lecture. Further, we saw that in general we will have a free surface on which waves of small scales are continually falling over and thus dissipating energy. It does not

seem an exaggeration to call such an energy-dissipating flow "turbulent". This connection between energy non-conservation and turbulence has appeared in inviscid hydrodynamics before, namely, in the process of satisfying the jump conditions in nonlinear shallow water theory, where the loss of energy has been attributed to "turbulence" (Ref. 8).

Thus, with reference to (2.10), we tentatively define a barotropic flow to be turbulent in a space-time domain $\mathscr{D}$ if and only if

$$\int_{\mathscr{D}} \left[\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (u\varepsilon) + \nabla \cdot (uP)\right] dxdt < 0 \tag{3.1}$$

If we get equality instead of inequality in (3.1), we say that the flow is non-turbulent in $\mathscr{D}$. (A definition of turbulence which is tied more closely to the stability or instability of a flow will be given in equation (5.5).) The water wave problem is considered as the limit of the barotropic flow (2.8) as $\varepsilon \to 0$. In the case of an inviscid fluid with a thermal, as well as caloric, equation of state, one requires energy conservation, and the appearance of turbulence would manifest itself in the creation of entropy.

The definition of turbulence (3.1) (and the definition (5.5) that supersedes it) allows us to proceed in an axiomatic fashion, without reference to scales, since whether a quantity is zero or negative is not a matter of scale. It may be objected that this advantage of the formulation is more apparent than real, since there is always some viscosity, and thus some energy dissipation. But we come again to the matter of choice we make at the beginning regarding the scale of phenomena we wish to follow. Assuming that a viscosity, no matter how small, will always make the flow regular on a sufficiently small scale and that we care to follow the details of the flow on that scale, we may regard the flow as viscous and non-turbulent. On the other hand, if the energy dissipation is significant, whereas the energy loss for a flow with the given viscosity and flow quantities varying on the large scale on which we prefer to study the motion is insignificant, we are justified in proceeding asymptotically, treating the flow as inviscid and turbulent. In this case the energy loss appears to be due to the presence of "eddy viscosity", in the language of the phenomenological approach to turbulence.

4. Helmholtz Instability

However we may define turbulence, the problem of well-posedness in the sense of Hadamard remains. As we have seen, this problem arises for flows with non-Lipschitzian velocities. We know of no way to avoid this problem in the context of classical theory. For example, one has the problem of the Helmholtz instability for an inviscid incompressible fluid in which in the basic state two streams of the fluid with a plane interface have each a constant velocity which is discontinuous at the interface with the velocity discontinuity lying

in the plane. If the discontinuity of velocity is of magnitude 2U, one finds that initial perturbations on the interface of wave number k have, in the context of linear disturbance theory, an exponential growth rate in time proportional to $U\,|k|$, and the initial value problem is not well-posed in the sense of Hadamard (Ref. 1). For the rest of this paper, we will refer to the instability associated with non-Lipschitzian velocities as "Helmholtz instability".

Another example is given by a solution of the hyperbolic conservation laws (1.4a, b) with g = 0 and no density constraint, in other words, the equations of a perfectly compressible fluid. We rewrite the equations here:

$$\rho_t + \nabla \cdot (\rho u) = 0 \,. \tag{4.1a}$$

$$(\rho u)_t + \nabla \cdot (\rho u u) = 0 \,. \tag{4.1b}$$

We refer to (4.1) as the N-dimensional form of Burgers' equation. Our example is shown in Figure 1. One may start with

$$\rho_0 = \begin{cases} 1 & 1 < |z| < 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.2a}$$

and

$$u_0 = -\operatorname{sgn}(z)\, \vec{k}, \tag{4.2b}$$

as shown in part (a) of Figure 1. The subsequent solution is

$$\rho = \begin{cases} 1 & 1 - t < |z| < 2 - t,\ 0 < t < 1 \\ 0 & \text{otherwise}\ , \end{cases} \tag{4.3a}$$

$$u = -\operatorname{sgn}(z)\, \vec{k},\ 0 < t < 1, \tag{4.3b}$$

$$\rho = 2(t-1)\ \delta(z) + \begin{cases} 1 & 0 < |z| < 2 - t,\ 1 < t < 2 \\ 0 & \text{otherwise}\ , \end{cases} \tag{4.3c}$$

$$u = \begin{cases} -\vec{k} & z > 0,\ 1 < t < 2 \\ \vec{k} & z < 0,\ 1 < t < 2 \\ 0 & z = 0,\ 1 < t < 2, \end{cases} \tag{4.3d}$$

$$\rho = 2\delta(z),\ t > 2, \tag{4.3e}$$

$$u = 0,\ t > 2. \tag{4.3f}$$

Suppose initially the density field (4.2a) is unchanged, but the velocity field (4.2b) is slightly perturbed by the addition of a horizontal velocity,

so that

$$\tilde{u}_0 = - \operatorname{sgn}(z)\vec{k} + \begin{cases} -\frac{\vec{i}}{2}(x - id) & |x - id| < \frac{d}{2},\ z > 0 \\ -\frac{\vec{i}}{2}(x - (i + \tfrac{1}{2})d & |x - (i + \tfrac{1}{2})d| < \frac{d}{2},\ z < 0\ . \end{cases} \tag{4.4}$$

Here i is an integer, $-\infty < i < \infty$. At $t = 1$ the perturbed density field will be

$$\tilde{\rho} = \begin{cases} 2 & |x - id| < \frac{d}{4},\ 1 > z > 0 \\ 0 & \frac{d}{4} < |x - id| < \frac{d}{2},\ 1 > z > 0 \\ 2 & |x - (i + \tfrac{1}{2})d| < \frac{d}{4}, -1 < z < 0 \\ 0 & \frac{d}{4} < |x - (i + \tfrac{1}{2})d| < \frac{d}{2}, -1 < z < 0\ , \end{cases} \tag{4.5}$$

and subsequently the streams will pass through one another without collision. This situation is shown in part (b) of Figure 1.

To recapitulate to this point: We began with the notion of turbulence as representing a state of "complexity" in a flow. Recognizing "complexity" as an inherently asymptotic concept, we were led to consider the cases of "infinite complexity", namely, those of instability, as examples of turbulent flows. On the way we found that the Helmholtz instability played a central role. Since energy dissipation was often associated with Helmholtz instability, we found it convenient to relate the turbulent character of a flow to the status of energy conservation for the flow. (According to the definition (3.1), certain unstable flows will be classified as non-turbulent. An example is the "flow" obtained by taking the limit as $d \to 0$ of the energy-preserving flow in (4.4) and (4.5). This "flow" is energy-preserving but unstable. A more satisfactory categorization of turbulent flows will be given in Section 5.)

Now we should ask the reverse question: Are the flows we have characterized as "turbulent" in fact "complicated", in that they exhibit instability with respect to small perturbations? (Observe that the fact that we can write down the solution explicitly does not mean that it is stable.) For the case of Burgers' equation the answer seems fairly clear. We just gave an example of a solution of Burgers' equation which was unstable with respect to perturbations of dimension two or higher. On the other hand, had we regarded the flow described by (4.2) and (4.3) as a one-dimensional flow subject to one-dimensional perturbations, it appears that the flow would have been stable. For incompressible flows, we note that the initial conditions for the flow represented in Figure 1 of our last lecture (Ref. 7) look very much like those for the flow in part (a) of Figure 1 of this paper. However, perturbations like (4.4) on the

basic flow in the incompressible case will not have the effect on the density at $t = 1$ that they had in the perfectly compressible case, and it appears that the flow of Figure 1 of the last lecture is stable with respect to small perturbations in any number of dimensions, but is still classified as turbulent. In spite of this, generally when there is a free surface in an incompressible flow we would expect stability in one dimension, but not in two. For incompressible flows without a free surface, our expectation is that energy-nonconserving flows will also be unstable. Perhaps our characterization of certain stable flows whose solutions can be written down explicitly as turbulent will be regarded as unsatisfactory. However, these flows seem to be rather special and of limited interest. For the multitude of flows of interest, we anticipate that our characterization and instability will go hand-in-hand.

5. Modified Stability Criterion

The motivation for requiring stability for well-posed problems seems to be something like this: Real problems in nature are never described with perfect accuracy. There will always be some error in the conditions given for the problem, as well as in the laws of physics which provide the governing equations. Thus, if the slightest change in the conditions of the problem leads to a large change in the predictions from the data, no reliability can be ascribed to the predictions, since in reality there will be uncontrollable perturbations on the conditions.

The notion that the perturbations are "uncontrollable" gives us an opening for a modified criterion. By imposing initial and boundary data with a given degree of regularity on a problem, we are implying that the problem can be controlled on the scales of relevance for the imposed data. Presumably errors on a smaller scale cannot be controlled, and we interpret this to mean that they are random and have no correlation with one another. Thus, we introduce the element of statistics into the problem, and we weaken our concept of stability to that of "stochastic stability": The flow is stochastically stable if, with probability 1, the flow does not change as the magnitude of the stochastic perturbations goes to zero.

Thus, the flow of (4.2) and (4.3) is not stable, because no matter how small d is, we can find a perturbed flow given by (4.4) and (4.5). But we expect the flow to be stochastically stable, because we expect the particular perturbation given by (4.4) to occur with probability 0. (This implies that we have already chosen the statistics for the perturbations. We will clarify this below.) Each of the flows (4.4) and (4.5) is both stable and stochastically stable, although it is unstable with respect to perturbations in the velocity of magnitude d. Regarding the limit of (4.4) and (4.5) as $d \to 0$ as a flow, we anticipate that it is unstable and stochastically unstable.

As underlying statistics for the problem, we assume as a first approximation that fluid elements at different locations are constantly undergoing random walks, and we regard the solution of Burgers' equation (4.1) to be the limit attained as the strength of the random walks goes to zero. Since this limit has some similarity to the conception of an inviscid flow as the limit as $\nu \to 0$ of a viscous flow, let us point out the differences. The viscosity ν is thought of as a representation of the effect of molecular random walks over a mean free path which occur with such frequency that one is able to deal with the average of these walks. Accordingly, although there is an underlying statistical theory, the Navier-Stokes equations themselves are deterministic. In the case we consider now, we have already let $\nu \to 0$, and the system is subjected to larger, non-molecular fluctuations, whose strength is nonetheless very small. Thus, we look at a stochastic flow field in the limit as the stochastic processes vanish, but without averaging first.

As an indication of how we proceed, we recall the procedure whereby we have solved Burgers' equation approximately for a time interval τ (Ref. 7). Given (ρ,u) we found

$$F(x,v,0) = \rho(x)\,\delta\,(v-u(x)), \tag{5.1a}$$

solved

$$F_t + v \cdot \nabla F = 0\ , \qquad 0 < t < \tau\ , \tag{5.1b}$$

and determined

$$\tilde{\rho} = \int F(x,v,\tau)dv\ , \tag{5.2a}$$

$$\tilde{\rho}\tilde{u} = \int vF(x,v,\tau)dv\ . \tag{5.2b}$$

In our modified stochastic approach, we leave (5.1a) and (5.2) unchanged, but regard F, ρ, and u as stochastic variables. In place of (5.1b) we have

$$dF \equiv (F_t + v \cdot \nabla F)dt = -\nabla\cdot(\varepsilon dwF) \tag{5.3}$$

where dw is a Brownian motion and $\varepsilon(x,t)$ is a positive parameter. We then consider the limit of (5.1a), (5.3), and (5.2) as $\varepsilon \to 0\ \forall\ x,t$.

It is too early for us to speak with certainty of the regularity of solutions of Burgers' equation. However, the following sort of function space seems a likely candidate for the velocity field u when (5.1a), (5.3), and (5.2) yield a deterministic solution in the limit as $\varepsilon \to 0$. We require that the velocity field u be almost continuous (a.c.), according to the following definition: A function f is almost continuous (a.c.) with respect to an unsigned measure dm in a domain $\mathcal{D}$ contained in a unit ball B if and only if, for every $m > 0$, it is possible fo find another unsigned measure $d\tilde{m}$ such that

$$\int_{\mathfrak{D}} d\tilde{m} = \int_{\mathfrak{D}} dm - m \tag{5.4}$$

and f is continuous with respect to x on the support of $d\tilde{m}$. f is a.c. in a domain $\mathfrak{D}$ if f is a.c. in $\mathfrak{D} \cap B$, uniformly for every unit ball. (The notion of "almost continuous" functions is a generalization of the notion of "almost uniformly continuous" (a.u.c.) functions. The natural regularity associated with some nonlinear semi-groups which are contractive in L' is almost uniform continuity (Refs. 2, 5).) We are inclined to take the initial data u_0 to be a.c. with respect to the measure $\rho_0(x)dx$. For the subsequent solution, we may want $u(x,t)$ to be a.c. in space-time with respect to the measure $\rho dxdt$. (Or perhaps more suggestive, from the viewpoint of modern physics, is the measure $\rho u^2 dxdt$.)

If the solution to the problem in the limit as $\varepsilon \to 0$ is not deterministic, but is stochastic, then the definitions above will have to be modified. For example, by "flow" we will now mean a collection of deterministic flows, each with its own probability $p(\alpha)d\alpha$ labeled by a parameter α. By "stochastic stability" of a flow we will mean that, with probability 1, the probability distribution of flows does not change as the strength of the stochastic perturbations goes to zero. In place of the measures for a.c. functions used above, we will want almost continuity with respect to the measures $\rho_o(x)dxp(\alpha)d\alpha$, $\rho(x)dxdtp(\alpha)d\alpha$, and $\rho u^2 dxdtp(\alpha)d\alpha$. Finally, we alter the definition (3.1) of turbulence as follows: A barotropic flow is turbulent in a space-time domain $\mathfrak{D}$ if and only if

$$\int\int_{\mathfrak{D}} \left| \frac{\partial \varepsilon(\alpha)}{\partial t} + \nabla \cdot (u(\alpha)\varepsilon(\alpha)) + \nabla \cdot (u(\alpha)P(\alpha)) \right| dxdtp(\alpha)d\alpha < 0. \tag{5.5}$$

(With this definition, it appears that the flow obtained as the limit $d \to 0$ of (4.4) and (4.5) will be classified as "turbulent".)

6. Stochastic Evolution from Deterministic Data

At present we do not have any examples of solutions of Burgers' equation which are initially deterministic but which for $t > 0$ remain stochastic in the limit as $\varepsilon \to 0$ in (5.3). In fact, we do not thick this happens for Burgers' equation in one dimension. For higher dimensions, we leave open the possibility of such a development.

We do not see any serious breach of fundamental physical principles in allowing an initially deterministic system to evolve stochastically. Such a situation would seem to correspond to a number of cases of turbulent flow, in which the data are irreproducible, even though the statistics for the data may be determinate. Moreover, such a situation will not appall those trained in modern physics. (The theory might even be vaguely reminiscent of "hidden variables" theory.) What we do require of a theory is not that the flow be deterministic, but that the theory yield unambiguous statistics for the possible

outcomes.

Let us give an example of the sort of behavior one may expect, by considering an initial value problem for a single nonlinear ordinary differential equation of first order. (An extension of the results reported below to a system of ordinary differential equations is currently under way.) The case of turbulent flows would be an infinite-dimensional generalization of this. The example we give, while admittedly unrealistic, is perhaps not as far-fetched as one would think. We have found the lack of Lipschitz continuity of the velocity field to be of great relevance to the problems of stability, energy conservation, and turbulence. In solving Burgers' equation we use solutions of the collisionless Boltzmann equation (5.1b), whose characteristics satisfy the ordinary differential equations

$$\frac{dx}{dt} = v$$

the solutions of which, because of (5.1a), are concentrated (in a measure-theoretic sense) near $v = u(x)$.

Consider the problem

$$\frac{dx}{dt} = x^{1/3} \quad , \quad t > 0, \ x(0) = 0. \tag{6.1}$$

Solutions of (6.1) are

$$x(t) = \begin{cases} 0 & 0 \le t \le t_0 \\ \pm \left(\frac{2}{3}(t-t_0)\right)^{3/2} & t \ge t_0 \end{cases} \tag{6.2}$$

We get a valid solution for any value of $t_0 \ge 0$ and for either sign in (6.2).

Now consider the stochastic equation

$$dx = x^{1/3}\,dt + \varepsilon\,dw \ , \tag{6.3}$$

for $\varepsilon > 0$. The solution of (6.3) is unique in a probabilistic sense for any $\varepsilon > 0$ (Ref. 3). Next we consider the limit of (6.3) as $\varepsilon \to 0$. In this case, the argument is very simple. We can show that, for any $\delta > 0$, we have for ε small enough, but not zero, a finite probability independent of δ that a "particle" on the axis $x = 0$ will leave the axis in time δ , never to return. Then taking the limits as $\varepsilon \to 0$ and $\delta \to 0$ in turn, we conclude that the solutions of (6.2) with $t_0 > 0$ occur with probability 0. By symmetry, we get, in the limit $\varepsilon \to 0$, with probability ½

$$x(t) = \left(\frac{2}{3}\,t\right)^{3/2},$$

and with probability ½

$$x(t) = -\left(\frac{2}{3}\,t\right)^{3/2}.$$

At this point we are just beginning the hard analysis dealing with the

well-posing of Burgers' equation (4.1) in a stochastic sense and in suitable function spaces. Beyond that, we have similar tasks to perform for the generalized hydrodynamic equations (1.4) - (1.6). Such results as we have obtained are not worth reporting at this time.

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